

On the icosahedron: from two to three dimensions.

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Abstract

In his famous book [2], Felix Klein describes a complex variable for the quotients of the ordinary sphere by the finite groups of rotations and in particular for the most complex situation of the quotient by the symmetry group of the icosahedron. The purpose of this work and its sequels is to obtain similar results for the quotients of the three-dimensional sphere. Various properties of the group $SU(2)$ and of its representations are used to obtain explicit expressions for coordinates and the relations they satisfy.

1 Introduction

The study of the quotients of the two dimensional sphere by subgroups of the rotation group has a long story. The identification of the sphere with the complex projective sphere has linked this study to the one of binary forms and their invariant combinations (see in particular [1]), and it has been masterfully exposed in the work of Felix Klein [2] more than a century ago. It has nevertheless seen a renewed interest with its application to the classification of singularities of surfaces through the McKay correspondence [3, 4, 5].

The quotients of the three-dimensional sphere are characterized by the same subgroups of $SU(2)$ and were classified long ago [7]. The possible cosmological interest of such quotients, as proposed by Luminet et al. [6], renewed the interest for the functions defined on such quotients. In recent works, a description of the functions on the regular quotients of the three sphere has been reached [8, 9], but the algebra formed by these functions was not considered. This work aims at filling this gap, describing generators of these algebras and the relations they satisfy. The reverse problem of identifying from a description by generator and relations the algebra of functions of some quotient of the three sphere will be reserved for a subsequent work [10].

This work uses the fundamental fact that the functions on the quotients form multiplets of $SU(2)$. When considering polynomials in such functions, the use of Clebsch–Gordan coefficients allows to group them in irreducible representations of $SU(2)$. This produces either new multiplets of functions or relations.

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The next section will recall the procedure for the enumeration of the functions on the quotients. Then the main technical tool is introduced, which allows to relate operations in the three-dimensional realm to known invariant combinations in the two-dimensional one. This allows to give a general description of the generators of the algebra of functions. In some cases, one can use sections of non-trivial line bundles to obtain embeddings of the quotient in a projective space. Finally, I show how this structure is realized for every of the finite subgroups of $SU(2)$.

2 The functions.

The functions on the quotients of the three-sphere, classified by eigenspaces of the Laplacian, can be determined from the special structure of the three-sphere. Indeed, this sphere can be given the group structure of $SU(2)$ or the unit quaternion. Its symmetry group is the product of its right and left actions on itself. Functions on the sphere form a representation of the product of these two $SU(2)$ groups. Its decomposition in irreducible representations involves all the possible product of a representation of $SU(2)^L$ with the equivalent one of $SU(2)^R$.

The identification with $SU(2)$ is made clear by introducing complex variables:

$$p = \begin{pmatrix} s & -\bar{t} \\ t & \bar{s} \end{pmatrix} \quad (1)$$

with the constraint $s\bar{s} + t\bar{t} = 1$. With these variables, the action of the Lie generators of the groups $SU(2)^L$ and $SU(2)^R$ can be written as:

$$\begin{aligned} J_z^L &= \frac{1}{2}(s\partial_s + \bar{t}\partial_{\bar{t}} - t\partial_t - \bar{s}\partial_{\bar{s}}), \\ J_+^L &= s\partial_t - \bar{t}\partial_{\bar{s}}, \\ J_-^L &= t\partial_s - \bar{s}\partial_{\bar{t}}, \end{aligned} \quad (2)$$

$$\begin{aligned} J_z^R &= \frac{1}{2}(s\partial_s - \bar{t}\partial_{\bar{t}} + t\partial_t - \bar{s}\partial_{\bar{s}}), \\ J_+^R &= -s\partial_{\bar{t}} + t\partial_{\bar{s}}, \\ J_-^R &= -\bar{t}\partial_s + \bar{s}\partial_t. \end{aligned} \quad (3)$$

The quotients we consider in this work are quotient by a subgroup of one of the two $SU(2)$ subgroups, so that they are automatically regulars. If we suppose that the monodromy group is a subgroup of $SU(2)^L$, the functions on the quotients will appear in multiplets of $SU(2)^R$ and new functions can be obtained from a given one through the action of J_-^R or J_+^R . The highest weight vector in a representation of $SU(2)^R$ can be represented by a function of only s and t .

The functions on the quotients of the three-dimensional sphere by a subgroup Γ of $SU(2)^L$ can therefore be described as follows. They form multiplet of $SU(2)^R$ in which all elements can be deduced from the highest weight vector by the action of J_-^R . The possible highest weight vectors are functions of s and t invariant under Γ . The invariant functions are known since the work of Klein [2]. They are generated by three functions satisfying an algebraic relation and can all be made from the lowest degree one by the action of the Hessian

and the cross-product. In the sequel, we show how to go from this generation through the action of the lowering operator J_-^R to a direct generation by algebraic operations.

3 Clebsch–Gordan

Since the functions come in multiplets of $SU(2)$, the different products of functions in multiplets of spin j_1 and j_2 of $SU(2)$ belong to the tensor product $(j_1) \otimes (j_2)$, which can be decomposed in the sum of representations $(j_1 + j_2)$ down to $(|j_1 - j_2|)$.

We model the representations on the one by homogeneous polynomials of degree $2j$ in two variables u and v . The monomials $\langle j, m \rangle = u^{j+m}v^{j-m}$ are vectors with the eigenvalue m of L_z , but they are not normalized. However the representations of L_+ and L_- in this basis is simple with integer matrix elements:

$$L_+(\langle j, m \rangle) = (j - m)\langle j, m + 1 \rangle, \quad L_-(\langle j, m \rangle) = (j + m)\langle j, m - 1 \rangle. \quad (4)$$

This translates in simple formulas for the Clebsch–Gordan coefficients. Differential formulas have been given in [9] for the general case and do not involve any squared roots. In this work, we are mainly interested in the highest weight vectors of the representations appearing in the composition of two irreducible representations. We have, up to normalization:

$$\langle j_1 + j_2 - k, j_1 + j_2 - k \rangle = \sum_{l=0}^k (-1)^l \binom{l}{k} \langle j_1, j_1 - l \rangle \langle j_2, j_2 - (k - l) \rangle, \quad (5)$$

since L_+ annihilates the right hand side of this equation.

In the case of the functions on the sphere, the element $\langle j, j \rangle$ of a $SU(2)^R$ multiplet is a homogeneous polynomial of degree $2j$ in (s, t) and the other elements of the multiplet are obtained by the action of L_-^R .

$$(L_-^R)^l \langle j, j \rangle = \frac{(2j)!}{(2j - l)!} \langle j, j - l \rangle \quad (6)$$

In equation (5), we need the vectors $\langle j_1, j_1 \rangle$ up to $\langle j_1, j_1 - k \rangle$ and we therefore will multiply this equation by a factor $(2j_1 - l)!/(2j_1 - k)!$ to have the common factor $(2j_1)!/(2j_1 - k)!$. This additional factor can be obtained by a degree counting operator, since the left hand side of equation (6) is homogeneous of degree $2j - l$ in the variables s and t . The homogeneous degree is obtained with the differential operator $s\partial_s + t\partial_t$. Switching to alternative variables s_1 and t_1 and then back, the full factor is obtained through powers of the operator $D_1 = s\partial_{s_1} + t\partial_{t_1}$. Finally, the change to the variables s_1 and t_1 can be done before the actions of L_-^R if we substitute this operator by $L_1 = \bar{s}\partial_{t_1} - \bar{t}\partial_{s_1}$. We thus obtain the following expression for the vectors appearing in eq. (5):

$$\frac{(2j_1)!}{(2j_1 - k)!} \langle j_1, j_1 - l \rangle = (D_1^{l-k} L_1^k \langle j_1, j_1 \rangle)|_{s_1=s, t_1=t} \quad (7)$$

Using the corresponding representation for the elements $\langle j_2, j_2 - (l - k) \rangle$, with the operator L_2 and D_2 defined in terms of ∂_{s_2} and ∂_{t_2} , we transform formula (5) into:

$$\begin{aligned} & \frac{(2j_1)!(2j_2)!}{(2j_1 - k)!(2j_2 - k)!} \langle j_1 + j_2 - k, j_1 + j_2 - k \rangle \\ &= \sum_{l=0}^k (-1)^l \binom{l}{k} D_1^{l-k} L_1^k \langle j_1, j_1 \rangle D_2^k L_2^{l-k} \langle j_2, j_2 \rangle. \end{aligned} \quad (8)$$

In this formula, $\langle j_i, j_i \rangle$ is understood as a function of s_i and t_i and we left implicit the step of changing all remaining s_i and t_i variables to s and t . Now the differential operators L_2 and D_2 do not act on the variables in $\langle j_1, j_1 \rangle$ and their coefficients do not depend on the differentiation variable. This is also true of L_1 and D_1 so we are free to reorder the terms in (8). The sum on l is therefore the binomial expansion of $(D_1 L_2 - L_1 D_2)^k$ applied to a product of polynomials in (s_1, t_1) and (s_2, t_2) . Now

$$D_1 L_2 - L_1 D_2 = (s\bar{s} + t\bar{t})(\partial_{s_1} \partial_{t_2} - \partial_{t_1} \partial_{s_2}). \quad (9)$$

We obtain therefore that in the combination of multiplets of spin j_1 and j_2 with the spin $j_1 + j_2 - k$ appears a factor $(s\bar{s} + t\bar{t})^k$, which is 1 on the sphere. The same result could be obtained from the condition that the highest weight in the multiplet of spin $j_1 + j_2 - k$ is annihilated by L_+^R , but we obtained a more precise result. Expanding the result of equation (9) and returning to the only variables s and t shows that the highest weight vector can be obtained by

$$\frac{(2j_1 - k)!(2j_2 - k)!}{(2j_1)!(2j_2)!} \sum_{l=0}^k (-1)^l \binom{k}{l} (\partial_s^{k-l} \partial_t^l \langle j_1, j_1 \rangle) (\partial_s^l \partial_t^{k-l} \langle j_2, j_2 \rangle). \quad (10)$$

This is the precise definition of the covariants $(f, g)^k$ used in the studies of the nineteenth century [1], even the global factor is the same. This formula was used for the combination of representations of $SU(2)$ defined as binary homogeneous polynomials in [9].

4 Generators of the algebra of functions.

Now the description of the algebra of functions on the quotient becomes clear. We take a first multiplet M of functions in the representation of spin j of $SU(2)^R$, with a highest weight vector P . Their binary products belong to the representations of spin $2j, 2j - 2, \dots$. The multiplet with spin $2j - 2$ has Q , the Hessian of P as highest weight, from the result of the preceding section. Among the ternary products of the functions of M we can select the representation of spin $3j - 3$ of $SU(2)^R$, which can be obtained as a special component of the composition of the multiplet of spin j and $2j - 2$. It will have as highest weight R , the cross-product of P and Q .

Klein proved that every invariant polynomial can be expressed as a product of the polynomials P, Q and R . Every multiplet can therefore be obtained as the

highest spin component in the combination of p times the multiplet M , q times the multiplet of spin $2j - 2$ generated by Q and r times the multiplet generated by R . Since the multiplets generated by Q and R are formed from polynomials of degree 2 and 3 of the elements of the multiplet M , every multiplet appears as a polynomial in the elements of M , which are therefore coordinates for the quotient space. In the cases of the group C_n and D_{4n} , this description is not sufficient, since the first invariant polynomial $P = st$ has Hessian 1 in the case C_n and in the case D_{4n} , $P = s^2 t^2$ has a Hessian proportional to itself. The multiplet generated by Q , with a degree proportional to n has to be introduced independently.

Is is however not always necessary to use multiplets generated by invariants.

Indeed, let us consider a polynomial V in s and t which belongs to some one-dimensional representation of Γ . Since the actions of $SU(2)^R$ and Γ commute, the polynomials generated from V by the action of L_-^R are multiplied by the same factor under the action of Γ . The multiplet they form is not invariant under Γ , but the different transformations under Γ are projectively equivalent. We obtain an imbedding in a projective space of smaller dimension than the affine space whose coordinates are the invariant functions.

The definition of the quotient as an algebraic variety does not end with the identification of coordinates. We further need to describe the relations they satisfy. Every combination of the coordinates with a $SU(2)^R$ spin which do not correspond to any invariant of Γ must vanish. This provides for a rich set of relations. The difficult part will be to identify a sufficient set of relations from which all others can be deduced. This will be done in [10].

5 Explicit results.

5.1 Cyclic group \mathbb{Z}_n .

In this case, all representations of Γ are one-dimensional, so that we could use the doublets $(s, -\bar{t})$ and (t, \bar{s}) as projective coordinates. However, this does not give an embedding of the quotient, since the map is the Hopf fibration of the 3-sphere with image the 2-sphere represented as the complex projective line. If we start from the lowest degree invariant, we have three coordinates in the spin 1 representation of $SU(2)$, which satisfy a quadratic relation and correspond to the euclidian representation of the two-dimensional sphere, again the quotient of the three-dimensional sphere by the Hopf fibration.

The n dependent part comes from the second independent invariant $s^n + t^n$. It gives rise to a representation of spin $n/2$. The combination with the coordinates of the two-sphere in a spin 1 representation yield representations of spin $n/2 - 1$, $n/2$ and $n/2 + 1$. The representation of spin $n/2 - 1$ must be zero and correspond to constraints and the combination of spin $n/2$ gives the multiplet generated by $s^n - t^n$. In the case where n is odd, the representations of half integer spin of $SU(2)$ are pseudoreal and complex conjugation relates the multiplet generated by $s^n + t^n$ to the one generated by $s^n - t^n$. This means that the reality conditions relate the multiplet of spin $n/2$ and its combination with the 3 coordinates of the two-sphere.

5.2 Dihedral groups D_{4n} .

This case is completely similar to the preceding one, apart from the fact that st and $s^n + t^n$ ($s^n + it^n$ in the case where n is odd) are in non-trivial representations of D_{4n} . We therefore have a description as a part of the product of the 2-dimensional projective space and a n -dimensional projective space. The relations remain the same.

5.3 Tetrahedral group.

In this case, we have a polynomial of degree 4 which belongs to a non-trivial one-dimensional representation of the group. Klein introduced it as the polynomial with zeros on the vertices of the tetrahedron:

$$V = s^4 + t^4 + 2i\sqrt{3}s^2t^2. \quad (11)$$

It is not fully invariant, being multiplied by a cubic root of the unity under the action of Γ . It is part of a representation of spin 2 of $SU(2)^R$, which is of dimension 5, providing an embedding of the quotient space in 4 dimensional projective space. The lowest degree invariant polynomial is of degree 6:

$$P = st(s^4 - t^4), \quad (12)$$

It will generate a representation of spin 3 of dimension 7.

The lowest dimensional embedding translates in a much simpler description. Indeed the space of homogeneous polynomials of degree n has dimension proportional to n^4 in one case and to n^6 in the other. When decomposing this space in irreducible representations of $SU(2)^R$, we have on the order n representations of dimensions of order n so that the typical multiplicity of a representation is proportional to n^2 in one case and n^4 in the other. Identifying a fundamental set of representations becomes much more easy in the projective case.

Quadratic polynomials in a spin 2 representation decompose in spin 4, spin 2 and spin 0 representations. The spin 0 representation can be seen to be zero, while the spin 2 representation has a generator similar to the one of V apart from the substitution of i by $-i$. The spin 0 representation gives one quadratic relation on the coordinates. It therefore seems that we simply recovered the three dimensional sphere as a quadric in a four dimensional projective space, but in fact, the non trivial geometry comes from the reality conditions, which link the fundamental spin 2 with the spin 2 part in the quadratic polynomials. This is coherent with the fact that the projective coordinates are multiplied by a cubic root of unity under the action of the group Γ .

The cubic polynomials in the projective coordinates are fully invariant under Γ and decompose into the representation $j = 0$, $j = 2$, $j = 3$, $j = 4$ and $j = 6$. The $j = 2$ representation vanishes since it can be written as the spin 0 relation times the fundamental representation. The $j = 0$ polynomial gives a number which can be used to normalize the projective representation. The ambiguity in the projective coordinates are reduced from the multiplication by any non zero complex number to the multiplication by a cubic root of unity. The $j = 3$, $j = 4$

and $j = 6$ representations are descendents of the three fundamental invariants, which are of degree 6, 8 and 12.

Since all invariants of the tetrahedron group can be obtained as products of the three fundamental ones, every function on the quotient of the three-sphere can be obtained from a polynomial of degree $3n$ in the five projective coordinates.

5.4 Octahedron group.

In this case also, we can also work with a projective representation. The invariant polynomial P of the tetrahedral group defined in (12) is invariant up to a sign under the symmetry group of the octahedron. It generates a spin 3 representation of $SU(2)^R$ which will give embedding in a six dimensional projective space. In this case, the reality conditions can be satisfied inside the multiplet, so that we are in a real projective space.

The quadratic polynomials decompose into components of spin 6, 4, 2 and 0. The spin 6 and 4 components are the descendents of the first two invariants of the group, the spin 2 one is the set of relations and the spin 0 objects allows for the normalization of the projective coordinates.

Among the cubic polynomials appears a spin 6 representation generated by the third invariant of the tetrahedron group. The descendants of the third invariant of the octahedron group form a representation of spin 9 which appears first the polynomials of degree four in the projective coordinates.

5.5 Dodecahedron group.

In this case, we cannot find a non trivial one-dimensional representation. We therefore must start from the 13 functions in the multiplet of spin 6 generated by the fundamental invariant of degree twelve of the dodecahedron group.

As in the other cases, among the quadratic polynomials we can find the spin 10 representation generated by the invariant of degree twenty and the cubic polynomials have a part of spin 15 which is generated by the invariant of degree thirty. We therefore identified a set of coordinates, such that every function on the Poincaré sphere is a function of these coordinates. The difficulty lies in the identification of a fundamental set of equations satisfied by these coordinates. The quadratic functions belong to the representations of all even spin from 0 to 12. The absence of invariants of the dodecahedron group means that the spin 2, 4 and 8 representations must be zero. The spin 6 representation in the quadratic part must be related to the fundamental spin 6 representation. In fact, we can use only part of these quadratic relations to generate all of them. This will be detailed in our next work [10].

6 Conclusion.

I have shown that the algebra of functions on the quotients of the sphere is generated by one or two $SU(2)$ multiplets of functions. The essential ingredient is the use of Clebsch–Gordan coefficients to combine products in multiplets: the

relation (5) links such operations on the functions of the three-sphere to equivalent ones on the holomorphic sections of bundles on the two-sphere, reducing to the situation studied by Klein.

If it is easy to identify relations from the multiplets which do not correspond to any invariant of Γ , it is more difficult to identify a sufficient sets of relations. In a companion paper, I will show that the quadratic relations are sufficient to generate all relations.

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